

Determinant and inverse of join matrices on two sets

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In honor of Abraham Berman, Moshe Goldberg, and Raphael Loewy

Abstract Let (P, \preceq) be a lattice and f a complex-valued function on P . We define meet and join matrices on two arbitrary subsets X and Y of P by $(X, Y)_f = (f(x_i \wedge y_j))$ and $[X, Y]_f = (f(x_i \vee x_j))$ respectively. Here we present expressions for the determinant and the inverse of $[X, Y]_f$. Our main goal is to cover the case when f is not semimultiplicative since the formulas presented earlier for $[X, Y]_f$ cannot be applied in this situation. In cases when f is semimultiplicative we obtain several new and known formulas for the determinant and inverse of $(X, Y)_f$ and the usual meet and join matrices $(S)_f$ and $[S]_f$. We also apply these formulas to LCM, MAX, GCD and MIN matrices, which are special cases of join and meet matrices.

Key words and phrases: Join matrix, Meet matrix, Semimultiplicative function, LCM matrix, GCD matrix, MAX matrix

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1 Introduction

Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having $f((x_i, x_j))$, the image of the greatest common divisor of x_i and x_j , as its ij entry. Analogously, let $[S]_f$ denote the $n \times n$ matrix having $f([x_i, x_j])$, the image of the least common multiple of x_i and x_j , as its ij entry. That is, $(S)_f = (f((x_i, x_j)))$ and $[S]_f = (f([x_i, x_j]))$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrices on S associated with f , respectively. The set S is said to be GCD-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$, and the set S is said to be factor-closed if it contains every divisor of x for any $x \in S$. Clearly every factor-closed set is GCD-closed but the converse does not hold.

In 1875 Smith [30] calculated $\det(S)_f$ when S is factor-closed and $\det[S]_f$ in a more special case. Since then a large number of results on GCD and LCM matrices have been presented in the literature. See, for example [2, 5, 6, 7, 9, 10, 11, 12, 13, 14, 22]. Haukkanen [8] generalized the concept of a GCD matrix into a meet matrix and later Korkee and Haukkanen [18] did the same with the concepts of LCM and join matrices. These generalizations happen as follows.

Let (P, \preceq) be a locally finite lattice, let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of P and let f be a complex-valued function on P . The $n \times n$ matrix $(S)_f = (f(x_i \wedge x_j))$ is called the meet matrix on S associated with f and the $n \times n$ matrix $[S]_f = (f(x_i \vee x_j))$ is called the join matrix on S associated with f . If $(P, \preceq) = (\mathbb{Z}^+, |)$, then meet and join matrices become respectively ordinary GCD and LCM matrices. However, some additional assumptions regarding the lattice (P, \preceq) are still needed and we analyse these in Section 2.

The properties of meet and join matrices have been studied by many authors (see, e.g., [3, 8, 10, 15, 16, 18, 20, 21, 23, 25, 27, 28]). Haukkanen [8] calculated the determinant of $(S)_f$ on an arbitrary set S and obtained the inverse of $(S)_f$ on a lower-closed set S and Korkee and Haukkanen [17] obtained the inverse of $(S)_f$ on a meet-closed set S . Korkee and Haukkanen [18] present, among others, formulas for the determinant and inverse of $[S]_f$ on meet-closed, join-closed, lower-closed and upper-closed sets S .

Most recently, Altinisik, Tuglu and Haukkanen [4] generalized the concepts of meet and join matrices and defined meet and join matrices on two sets. Later these matrices were also treated in [19]. Next we present the same definitions.

Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be two subsets of P . We define the meet matrix on X and Y with respect to f as $(X, Y)_f = (f(x_i \wedge y_j))$. In particular, when $S = X = Y = \{x_1, x_2, \dots, x_n\}$, we have $(S, S)_f = (S)_f$. Analogously, we define the join matrix on X and Y with respect to f as $[X, Y]_f = (f(x_i \vee y_j))$. In particular, $[S, S]_f = [S]_f$.

In [4] the authors presented formulas for the determinant and the inverse of the matrix $(X, Y)_f$. Applying these formulas they derived similar formulas for the matrix $[X, Y]_{1/f}$ with respect to semimultiplicative functions f with $f(x) \neq 0$ for all $x \in P$. The cases when f is not semimultiplicative or $f(x) = 0$ for some $x \in P$, however, were excluded from the examination. In this paper we give formulas that can also be used in these circumstances. We go through the same examinations presented in [4] but this time dually from the point of view of the matrix $[X, Y]_f$. That is we present expressions for the determinant and the inverse of $[X, Y]_f$ on arbitrary sets X and Y . In the case when $X = Y = S$ we obtain a determinant formula for $[S]_f$ and a formula for the inverse of $[S]_f$ on arbitrary set S . We also derive formulas for the special cases when S is join-closed and upper-closed up to $\vee S$. Similar kind of determinant formulas for $(S)_f$ and $[S]_f$ have already been presented in [18], although they were obtained and presented by a different approach. By setting $(P, \preceq) = (\mathbb{Z}^+, |)$ we obtain corollaries for LCM matrices, and as another special case we also consider MAX and MIN matrices. In case when $(P, \preceq) = (\mathbb{Z}, \leq)$, where \leq is the natural ordering of the integers, the MAX and MIN matrices of the set S are the matrices $[S]_f$ and $(S)_f$ respectively. MAX and MIN matrices have not been addressed before in this context.

2 Preliminaries

In the preceding section we defined the concept of GCD-closed set. Similarly, the set S is said to be LCM-closed if $[x_i, x_j] \in S$ whenever $x_i, x_j \in S$. Since the lattice $(\mathbb{Z}^+, |)$ does not have a greatest element, we need to define the dual concept for factor-closed set in a more special manner.

Definition 2.1. Let $\text{lcm } S = [x_1, x_2, \dots, x_n]$, the least common multiple of all the elements in S . The set S is said to be multiple-closed up to $\text{lcm } S$ if $x \in S$ whenever $y \in S$, $y | x$ and $x | \text{lcm } S$. In addition, let

$$M_S = \{y \in \mathbb{Z}^+ \mid y | \text{lcm } S \text{ and } x_i | y \text{ for some } x_i \in S\} = \bigcup_{i=1}^n [x_i, \text{lcm } S],$$

where

$$[x_i, \text{lcm } S] = \{y \in \mathbb{Z}^+ \mid x_i | y \text{ and } y | \text{lcm } S\}.$$

Again, if S is multiple-closed up to $\text{lcm } S$, then it is also LCM-closed, but an LCM-closed set is not necessarily multiple-closed up to $\text{lcm } S$. Obviously the set M_S is multiple-closed up to $\text{lcm } S$, but the semilattice $(M_S, |)$ also has the advantage of having the greatest element over the lattice $(\mathbb{Z}^+, |)$. Next we need corresponding definitions for a more general case.

Let (P, \preceq) be a lattice. The set $S \subseteq P$ is said to be lower-closed (resp. upper-closed) if for every $x, y \in P$ with $x \in S$ and $y \preceq x$ (resp. $x \preceq y$),

we have $y \in S$. The set S is said to be meet-closed (resp. join-closed) if for every $x, y \in S$, we have $x \wedge y \in S$ (resp. $x \vee y \in S$).

If every principal order filter of the lattice (P, \preceq) is finite, the methods presented in the following sections can be applied to the lattice (P, \preceq) directly. If the lattice (P, \preceq) does not satisfy this property (which is the case when, for example, $P = \mathbb{Z}^+$ and $\preceq = |$), it is always possible to carry out the following procedures in an appropriate subsemilattice of (P, \preceq) . The most straightforward method is to generalize Definition 2.1, which is done as follows.

Definition 2.2. Let $\vee S = x_1 \vee x_2 \vee \cdots \vee x_n$. The set S is said to be upper-closed up to $\vee S$ if $x \in S$ whenever $y \in S$, $y \preceq x$ and $x \preceq \vee S$. In addition, let

$$P_S = \{y \in L \mid y \preceq \vee S \text{ and } x_i \preceq y \text{ for some } x_i \in S\} = \bigcup_{i=1}^n [x_i, \vee S],$$

where

$$[x_i, \vee S] = \{y \in L \mid x_i \preceq y \text{ and } y \preceq \vee S\}.$$

Another possibility would be to restrict our consideration to $(\langle S \rangle, \preceq)$, the join-subsemilattice of (P, \preceq) generated by the set S . Usually this would also reduce the number of computations needed. For example, the values of the Möbius function of $(\langle S \rangle, \preceq)$ are often much easier to calculate than the values of the Möbius function of (P_S, \preceq) (see [1, Section IV.1]). And if we consider S as a subset of the meet-subsemilattice generated by itself, the set S is meet-closed iff it is lower-closed. Similarly, the terms join-closed and upper-closed coincide in the join-subsemilattice $(\langle S \rangle, \preceq)$. This is another benefit of restricting to $(\langle S \rangle, \preceq)$. This method is not, however, very convenient when considering the lattice $(\mathbb{Z}^+, |)$. The Möbius function of $(\langle S \rangle, \preceq)$, where $S \subset \mathbb{Z}^+$, has often very little in common with the number-theoretic Möbius function, which would likely cause confusion. For this reason we give our formulas in a form that fits both for the types of lattices defined in Definition 2.2 and for the lattice $(\langle S \rangle, \preceq)$.

Let (P, \preceq) be a locally finite lattice, and let f be a complex-valued function on P . Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ be two subsets of P . Let the elements of X and Y be arranged so that $x_i \preceq x_j \Rightarrow i \leq j$. Let $D = \{d_1, d_2, \dots, d_m\}$ be any subset of P containing the elements $x_i \vee y_j$, $i, j = 1, 2, \dots, n$. Let the elements of D be arranged so that $d_i \preceq d_j \Rightarrow i \leq j$. Then we define the function $\Psi_{D,f}$ on D inductively as

$$\Psi_{D,f}(d_k) = f(d_k) - \sum_{d_k \prec d_v} \Psi_{D,f}(d_v) \quad (2.1)$$

or equivalently

$$f(d_k) = \sum_{d_k \preceq d_v} \Psi_{D,f}(d_v). \quad (2.2)$$

Then

$$\Psi_{D,f}(d_k) = \sum_{d_k \preceq d_v} f(d_v) \mu_D(d_k, d_v), \quad (2.3)$$

where μ_D is the Möbius function of the poset (D, \preceq) (see [31, 3.7.2 Proposition]).

If D is join-closed, then

$$\Psi_{D,f}(d_k) = \sum_{\substack{d_k \preceq z \\ d_t \not\preceq z \\ k < t}} \sum_{z \preceq w \preceq s} f(w) \mu_{P_D}(z, w), \quad (2.4)$$

where μ_{P_D} is the Möbius function of (P_D, \preceq) , and if D is upper-closed up to $\vee D$, then

$$\Psi_{D,f}(d_k) = \sum_{d_k \preceq d_v} f(d_v) \mu_{P_D}(d_k, d_v), \quad (2.5)$$

where μ_{P_D} is the Möbius function of (P_D, \preceq) . The proofs of formulas (2.4) and (2.5) are dual to the proofs presented in [8].

Remark 2.1. If D is join-closed, then $D = \langle D \rangle$ and D is trivially upper-closed subset of $\langle D \rangle$ in $(\langle D \rangle, \preceq)$. Thus in this case we could also replace the μ_D in (2.3) and the μ_{P_D} in (2.5) by $\mu_{\langle D \rangle}$.

If $(P, \preceq) = (\mathbb{Z}^+, |)$ and D is multiple-closed up to $\text{lcm } D$, then $\mu_D(d_k, d_v) = \mu(d_v/d_k)$ (see [24, Chapter 7]), where μ is the number-theoretic Möbius function. In addition, for every $a \in \mathbb{Z}^+$ and arithmetical function f we may define another arithmetical function f_a , where

$$f_a(n) = f(an)$$

for every $n \in \mathbb{Z}^+$. Now from (2.3) we get

$$\begin{aligned} \Psi_{D,f}(d_k) &= \sum_{d_k | d_v} f(d_v) \mu\left(\frac{d_v}{d_k}\right) = \sum_{a | \frac{\text{lcm } D}{d_k}} f(d_k a) \mu(a) \\ &= \sum_{a | \frac{\text{lcm } D}{d_k}} (f_{d_k} \mu)(a) = [\zeta * (f_{d_k} \mu)]\left(\frac{\text{lcm } D}{d_k}\right), \end{aligned} \quad (2.6)$$

where $*$ is the Dirichlet convolution of arithmetical functions.

Let $E(X) = (e_{ij}(X))$ and $E(Y) = (e_{ij}(Y))$ denote the $n \times m$ matrices defined by

$$e_{ij}(X) = \begin{cases} 1 & \text{if } x_i \preceq d_j, \\ 0 & \text{otherwise,} \end{cases} \quad (2.7)$$

and

$$e_{ij}(Y) = \begin{cases} 1 & \text{if } y_i \preceq d_j, \\ 0 & \text{otherwise} \end{cases} \quad (2.8)$$

respectively. Clearly $E(X)$ and $E(Y)$ also depend on D but for the sake of brevity D is omitted from the notation. We also denote

$$\Lambda_{D,f} = \text{diag}(\Psi_{D,f}(d_1), \Psi_{D,f}(d_2), \dots, \Psi_{D,f}(d_m)). \quad (2.9)$$

3 A structure theorem

In this section we give a factorization of the matrix $[X, Y]_f = (f(x_i \vee y_j))$. A large number of similar factorizations is presented in the literature, for example in [16] the matrix $[S]_f$ is factorized in case when S is join-closed. The idea of this kind of factorization may be considered to originate from Pólya and Szegő [26].

Theorem 3.1.

$$[X, Y]_f = E(X)\Lambda_{D,f}E(Y)^T. \quad (3.1)$$

Proof. By (2.2) the ij entry of $[X, Y]_f$ is

$$f(x_i \vee y_j) = \sum_{x_i \vee y_j \preceq d_v} \Psi_{D,f}(d_v). \quad (3.2)$$

Now, applying (2.7), (2.8) and (2.9) to (3.2) we obtain Theorem 3.1. \square

Remark 3.1. The sets X and Y could be allowed to have distinct cardinalities in Theorems 3.1 and 6.1. However, in other results we must assume that these cardinalities coincide.

4 Determinant formulas

In this section we derive formulas for determinants of join matrices. In Theorem 4.1 we present an expression for $\det[X, Y]_f$ on arbitrary sets X and Y . Taking $X = Y = S = \{x_1, x_2, \dots, x_n\}$ in Theorem 4.1 we obtain a formula for the determinant of usual join matrices $[S]_f$ on arbitrary set S , and further taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we obtain a formula for the determinant of LCM matrices on arbitrary set S . In Theorems 4.2 and 4.3 respectively, we calculate $\det[S]_f$ when S is join-closed and upper-closed up to $\vee S$. Formulas similar to Theorems 4.2 and 4.3 but by different approach and notations are given in [18].

Theorem 4.1. (i) If $n > m$, then $\det[X, Y]_f = 0$.

(ii) If $n \leq m$, then

$$\begin{aligned} \det[X, Y]_f &= \sum_{1 \leq k_1 < k_2 < \dots < k_n \leq m} \det E(X)_{(k_1, k_2, \dots, k_n)} \det E(Y)_{(k_1, k_2, \dots, k_n)} \\ &\quad \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_n}). \end{aligned} \quad (4.1)$$

Proof. By Theorem 3.1

$$\det[X, Y]_f = \det \left(E(X)\Lambda_{D,f}E(Y)^T \right). \quad (4.2)$$

Thus by the Cauchy-Binet formula we obtain Theorem 4.1. \square

Theorem 4.2. *If S is join-closed, then*

$$\begin{aligned}\det[S]_f &= \prod_{v=1}^n \Psi_{S,f}(x_v) = \prod_{v=1}^n \sum_{x_v \preceq x_t} f(x_t) \mu_S(x_v, x_t) \\ &= \prod_{v=1}^n \sum_{\substack{x_v \preceq z \\ x_t \not\preceq z \\ v < t}} \sum_{z \preceq w \preceq \vee S} f(w) \mu_{P_S}(z, w).\end{aligned}\quad (4.3)$$

Proof. We take $X = Y = S$ in Theorem 4.1. Since S is join-closed, we may further take $\langle D \rangle = D = S$. Then $m = n$ and $\det E(S)_{(k_1, k_2, \dots, k_n)} = \det E(S)_{(1, 2, \dots, n)} = 1$ and so we obtain the first equality in (4.3). The second equality follows from Remark 2.1 and the third from (2.4). \square

Remark 4.1. Theorem 4.2 can also be proved by taking $X = Y = S$ and $D = S$ in Theorem 3.1.

Example 4.1. Let $(P, \preceq) = (\mathbb{Z}, \leq)$, where \leq is the natural ordering of the set of integers, and let $S = \{x_1, x_2, \dots, x_n\} \subset \mathbb{Z}$, where $x_1 < x_2 < \dots < x_n$. Let $t \in \mathbb{C}$ and $f : \mathbb{Z} \rightarrow \mathbb{C}$ be such function that $f(k) = k + t$ for all $k \in \mathbb{Z}$. Since the lattice (\mathbb{Z}, \leq) is a chain, the set S is trivially both meet and join-closed. Now it follows from Theorem 4.2 that the determinant of the MAX matrix $[S]_f$ is

$$\begin{aligned}\det[S]_f &= \prod_{v=1}^n \sum_{x_v \preceq x_t} f(x_t) \mu_S(x_v, x_t) \\ &= (f(x_1) - f(x_2))(f(x_2) - f(x_3)) \cdots (f(x_{n-1}) - f(x_n))f(x_n) \\ &= (x_1 - x_2)(x_2 - x_3) \cdots (x_{n-1} - x_n)(x_n + t).\end{aligned}\quad (4.4)$$

Theorem 4.3. *If S is upper-closed up to $\vee S$, then*

$$\det[S]_f = \prod_{v=1}^n \Psi_{S,f}(x_v) = \prod_{v=1}^n \sum_{x_v \preceq x_u} f(x_u) \mu(x_v, x_u). \quad (4.5)$$

Proof. The first equality in (4.5) follows from (4.3). The second equality follows from (2.5). \square

Example 4.2. Let (P, \preceq) , S and f be as in Example 4.1 and let $x_i = x_1 + (i - 1)$ for every $x_i \in S$. Now the set S is clearly upper-closed up to $x_n = x_1 + (n - 1)$ and from Theorem 4.3 we get the determinant of the MAX matrix $[S]_f$ as

$$\begin{aligned}\det[S]_f &= \prod_{v=1}^n \sum_{x_v \preceq x_t} f(x_t) \mu(x_v, x_t) \\ &= (f(x_1) - f(x_2))(f(x_2) - f(x_3)) \cdots (f(x_{n-1}) - f(x_n))f(x_n) \\ &= (-1)^{n-1}(x_n + t).\end{aligned}\quad (4.6)$$

Corollary 4.1. *Let $(P, \preceq) = (\mathbb{Z}^+, |)$, let S be an LCM-closed set of distinct positive integers, and let f be an arithmetical function. Then the determinant of the LCM matrix $[S]_f$ is*

$$\det[S]_f = \prod_{v=1}^n \sum_{\substack{x_v | z | \text{lcm } S \\ x_t \uparrow z \\ v < t}} [\zeta * (f_z \mu)] \left(\frac{\text{lcm } S}{z} \right). \quad (4.7)$$

Corollary 4.2. *Let $(P, \preceq) = (\mathbb{Z}^+, |)$, let S be a set of distinct positive integers which is multiple-closed up to $\text{lcm } S$, and let f be an arithmetical function. Then*

$$\det[S]_f = \prod_{v=1}^n [\zeta * (f_{x_v} \mu)] \left(\frac{\text{lcm } S}{x_v} \right). \quad (4.8)$$

5 Inverse formulas

In this section we derive formulas for inverses of join matrices. In Theorem 5.1 we present an expression for the inverse of $[X, Y]_f$ on arbitrary sets X and Y , and in Theorem 5.2 we present an expression for the inverse of $[S]_f$ on arbitrary set S . Taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we obtain a formula for the inverse of LCM matrices on arbitrary set S . Such formulas for the inverse of join or LCM matrices on an arbitrary set have not previously been presented in the literature. In Theorem 5.3 we calculate the inverse of $[S]_f$ on join-closed set S and in Theorem 5.4 we cover the case in which S is upper-closed up to $\vee S$.

Theorem 5.1. *Let $X_i = X \setminus \{x_i\}$ and $Y_i = Y \setminus \{y_i\}$ for $i = 1, 2, \dots, n$. If $[X, Y]_f$ is invertible, then the inverse of $[X, Y]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where*

$$b_{ij} = \frac{(-1)^{i+j}}{\det[X, Y]_f} \sum_{1 \leq k_1 < k_2 < \dots < k_{n-1} \leq m} \det E(X_j)_{(k_1, k_2, \dots, k_{n-1})} \det E(Y_i)_{(k_1, k_2, \dots, k_{n-1})} \\ \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \quad (5.1)$$

Proof. It is well known that

$$b_{ij} = \frac{\alpha_{ji}}{\det[X, Y]_f}, \quad (5.2)$$

where α_{ji} is the cofactor of the ji -entry of $[X, Y]_f$. It is easy to see that $\alpha_{ji} = (-1)^{i+j} \det[X_j, Y_i]_f$. By Theorem 4.1 we see that

$$\det[X_j, Y_i]_f = \sum_{1 \leq k_1 < k_2 < \dots < k_{n-1} \leq m} \det E(X_j)_{(k_1, k_2, \dots, k_{n-1})} \det E(Y_i)_{(k_1, k_2, \dots, k_{n-1})} \\ \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \quad (5.3)$$

Combining the above equations we obtain Theorem 5.1. \square

Theorem 5.2. Let $S_i = S \setminus \{x_i\}$ for $i = 1, 2, \dots, n$. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \frac{(-1)^{i+j}}{\det[S]_f} \sum_{1 \leq k_1 < k_2 < \dots < k_{n-1} \leq m} \det E(S_j)_{(k_1, k_2, \dots, k_{n-1})} \det E(S_i)_{(k_1, k_2, \dots, k_{n-1})} \times \Psi_{D,f}(d_{k_1}) \Psi_{D,f}(d_{k_2}) \cdots \Psi_{D,f}(d_{k_{n-1}}). \quad (5.4)$$

Proof. Taking $X = Y = S$ in Theorem 5.1 we obtain Theorem 5.2. \square

Theorem 5.3. Suppose that S is join-closed. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \sum_{k=1}^n \frac{(-1)^{i+j}}{\Psi_{S,f}(x_k)} \det E(S_i^k) \det E(S_j^k), \quad (5.5)$$

where $E(S_i^k)$ is the $(n-1) \times (n-1)$ submatrix of $E(S)$ obtained by deleting the i th row and the k th column of $E(S)$, or

$$b_{ij} = \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu_S(x_k, x_i) \mu_S(x_k, x_j)}{\Psi_{S,f}(x_k)}, \quad (5.6)$$

where μ_S is the Möbius function of the poset (S, \preceq) .

Proof. Since S is join-closed, we may take $D = S$. Then $E(S)$ is a square matrix with $\det E(S) = 1$. Further, $E(S)$ is the matrix associated with the zeta function of the finite poset (S, \preceq) . Thus the inverse of $E(S)$ is the matrix associated with the Möbius function of (S, \preceq) , that is, if $U = (u_{ij})$ is the inverse of $E(S)$, then $u_{ij} = \mu_S(x_i, x_j)$, see [1, p. 139]. On the other hand, $u_{ij} = \beta_{ij} / \det E(S) = \beta_{ij}$, where β_{ij} is the cofactor of the ij -entry of $E(S)$. Here $\beta_{ij} = (-1)^{i+j} \det E(S_i^j)$. Thus

$$(-1)^{i+j} \det E(S_i^j) = \mu_S(x_i, x_j). \quad (5.7)$$

Now we apply Theorem 5.2 with $D = S$. Then $m = n$, and using formulas (4.3) and (5.7) we obtain Theorem 5.3. \square

Remark 5.1. Equation (5.6) can also be proved by taking $X = Y = S$ and $D = S$ in Theorem 3.1 and then applying the formula $[S]_f^{-1} = (E(S)^T)^{-1} \Lambda_{S,f}^{-1} E(S)^{-1}$.

Example 5.1. Let (P, \preceq) , f and S be as in Example 4.1. If $t \neq -x_n$, then the matrix $[S]_f$ is invertible and the inverse of $[S]_f$ is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \frac{1}{x_1 - x_2} & \text{if } i = j = 1, \\ \frac{1}{x_{i-1} - x_i} + \frac{1}{x_i - x_{i+1}} & \text{if } 1 < i = j < n, \\ \frac{1}{x_{n-1} - x_n} + \frac{1}{x_n + t} & \text{if } i = j = n, \\ \frac{1}{|x_i - x_j|} & \text{if } |i-j| = 1. \end{cases}$$

Theorem 5.4. Suppose that S is upper-closed up to $\vee S$. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the $n \times n$ matrix $B = (b_{ij})$ with

$$b_{ij} = \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu(x_k, x_i) \mu(x_k, x_j)}{\Psi_{S,f}(x_k)}, \quad (5.8)$$

where μ is the Möbius function of (P, \preceq) .

Proof. Since S is upper-closed up to $\vee S$, we have $\mu_S = \mu$ on S , (apply [1, Proposition 4.6]). Thus Theorem 5.4 follows from Theorem 5.3. \square

Example 5.2. Let (P, \preceq) , f and S be as in Example 4.2. If $t \neq -x_n$, then the matrix $[S]_f$ is invertible and the inverse of $[S]_f$ is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i - j| > 1, \\ -1 & \text{if } i = j = 1, \\ -2 & \text{if } 1 < i = j < n, \\ -1 + \frac{1}{x_n + t} & \text{if } i = j = n, \\ 1 & \text{if } |i - j| = 1. \end{cases}$$

Corollary 5.1. Let S be a set of distinct positive integers which is multiple-closed up to $\text{lcm } S$, and let f be an arithmetical function. If the LCM matrix $[S]_f$ is invertible, then its inverse is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \sum_{x_k | (x_i, x_j)} \frac{\mu(x_i/x_k) \mu(x_j/x_k)}{[\zeta * (f_{x_k} \mu)] \left(\frac{\text{lcm } S}{x_k} \right)}. \quad (5.9)$$

Here μ is the number-theoretic Möbius function.

6 Formulas for meet matrices

Let f be a complex-valued function on P . We say that f is a semimultiplicative function if

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \quad (6.1)$$

for all $x, y \in P$ (see [18]).

The notion of a semimultiplicative function arises from the theory of arithmetical functions. Namely, an arithmetical function f is said to be semimultiplicative if $f(r)f(s) = f((r, s))f([r, s])$ for all $r, s \in \mathbb{Z}^+$. For semimultiplicative arithmetical functions reference is made to the book by Sivaramakrishnan [29], see also [9]. Note that a semimultiplicative arithmetical function f with $f(1) \neq 0$ is referred to as a quasimultiplicative arithmetical function. Quasimultiplicative arithmetical functions with $f(1) = 1$ are the usual multiplicative arithmetical functions.

In this section we show that meet matrices $(X, Y)_f$ with respect to semimultiplicative functions f possess properties similar to those given for join matrices $[X, Y]_f$ with respect to arbitrary functions f in Sections 3, 4 and 5. Since there already are several formulas for the determinant and the inverse of the matrix $(X, Y)_f$ (see [4] and [19]), the motivation in deriving new formulas probably needs clarification. The formulas of this section are especially useful when considering the matrix $(S)_f$, where the set S is either join-closed or upper closed up to $\vee S$. That is, because in this case the formulas of this section result in shorter and simpler calculations. *Throughout this section f is a semimultiplicative function on P such that $f(x) \neq 0$ for all $x \in P$.*

Theorem 6.1.

$$(X, Y)_f = \Delta_{X,f}[X, Y]_{1/f}\Delta_{Y,f} \quad (6.2)$$

or

$$(X, Y)_f = \Delta_{X,f}E(X)\Lambda_{D,1/f}E(Y)^T\Delta_{Y,f}, \quad (6.3)$$

where

$$\Delta_{X,f} = \text{diag}(f(x_1), f(x_2), \dots, f(x_n)) \quad (6.4)$$

and

$$\Delta_{Y,f} = \text{diag}(f(y_1), f(y_2), \dots, f(y_n)). \quad (6.5)$$

Proof. By (6.1) the ij -entry of $(X, Y)_f$ is

$$f(x_i \wedge y_j) = f(x_i) \frac{1}{f(x_i \vee y_j)} f(y_j). \quad (6.6)$$

We thus obtain (6.2), and applying Theorem 3.1 we obtain (6.3). \square

From (6.2) we obtain

$$\det(X, Y)_f = \left(\prod_{v=1}^n f(x_v) f(y_v) \right) \det[X, Y]_{1/f} \quad (6.7)$$

and

$$(X, Y)_f^{-1} = \Delta_{Y,f}^{-1}[X, Y]_{1/f}^{-1}\Delta_{X,f}^{-1}. \quad (6.8)$$

Now, using (6.7), (6.8) and the formulas of Sections 4 and 5 we obtain formulas for meet matrices.

We first present formulas for the determinant of meet matrices. In Theorem 6.2 we give a formula for $\det(X, Y)_f$ on arbitrary sets X and Y . This is an alternative expression that given in [4]. In Theorems 6.3 and 6.4 respectively, we calculate $\det(S)_f$ when S is join-closed and upper-closed up to $\vee S$.

Theorem 6.2. (i) If $n > m$, then $\det(X, Y)_f = 0$.

(ii) If $n \leq m$, then

$$\begin{aligned} \det(X, Y)_f &= \left(\prod_{v=1}^n f(x_v) f(y_v) \right) \left(\sum_{1 \leq k_1 < k_2 < \dots < k_n \leq m} \det E(X)_{(k_1, k_2, \dots, k_n)} \det E(Y)_{(k_1, k_2, \dots, k_n)} \right. \\ &\quad \left. \times \Psi_{D, 1/f}(d_{k_1}) \Psi_{D, 1/f}(d_{k_2}) \cdots \Psi_{D, 1/f}(d_{k_n}) \right). \end{aligned} \quad (6.9)$$

Theorem 6.3. If S is join-closed, then

$$\begin{aligned} \det(S)_f &= \prod_{v=1}^n f(x_v)^2 \Psi_{S, 1/f}(x_v) = \prod_{v=1}^n f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu_S(x_v, x_u)}{f(x_u)} \\ &= \prod_{v=1}^n f(x_v)^2 \sum_{\substack{x_v \preceq z \\ x_t \not\preceq z \\ v < t}} \sum_{z \preceq w \preceq \vee S} \frac{\mu(z, w)}{f(w)}. \end{aligned} \quad (6.10)$$

Example 6.1. Let $(P, \preceq) = (\mathbb{Z}, \leq)$, $t \in \mathbb{C}$ a complex number such that $t \neq -x_i$ for all $x_i \in S$ and $f(x_i) = x_i + t$ for all $x_i \in S$. Since (\mathbb{Z}, \leq) is a chain, the function f is trivially semimultiplicative. Now from Theorem 6.3 we get

$$\begin{aligned} \det(S)_f &= \prod_{v=1}^n f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu_S(x_v, x_u)}{f(x_u)} = \prod_{v=1}^n f(x_v)^2 \left(\frac{1}{f(x_v)} - \frac{1}{f(x_{v+1})} \right) \\ &= \prod_{v=1}^n f(x_v)^2 \frac{f(x_{v+1}) - f(x_v)}{f(x_v) f(x_{v+1})} \\ &= f(x_1)(f(x_2) - f(x_1))(f(x_3) - f(x_2)) \cdots (f(x_n) - f(x_{n-1})) \\ &= (x_1 + t)(x_2 - x_1)(x_3 - x_2) \cdots (x_n - x_{n-1}). \end{aligned}$$

Theorem 6.4. If S is upper-closed up to $\vee S$, then

$$\det(S)_f = \prod_{v=1}^n f(x_v)^2 \Psi_{S, 1/f}(x_v) = \prod_{v=1}^n f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu(x_v, x_u)}{f(x_u)}. \quad (6.11)$$

Example 6.2. Let $(P, \preceq) = (\mathbb{Z}, \leq)$, $S = \{x_1, x_1 + 1, x_1 + 2, \dots, x_1 + n - 1\}$, $t \in \mathbb{C}$ a complex number such that $t \neq -x_i$ for all $x_i \in S$ and $f(x_i) = x_i + t$ for all $x_i \in S$. Now it follows from Theorem 6.4 that

$$\begin{aligned} \det(S)_f &= \prod_{v=1}^n f(x_v)^2 \sum_{x_v \preceq x_u} \frac{\mu(x_v, x_u)}{f(x_u)} = \prod_{v=1}^n f(x_v)^2 \left(\frac{1}{f(x_v)} - \frac{1}{f(x_{v+1})} \right) \\ &= \prod_{v=1}^n f(x_v)^2 \frac{f(x_{v+1}) - f(x_v)}{f(x_v) f(x_{v+1})} \\ &= f(x_1)(f(x_2) - f(x_1))(f(x_3) - f(x_2)) \cdots (f(x_n) - f(x_{n-1})) \\ &= (x_1 + t)(-1)^{n-1}. \end{aligned}$$

Corollary 6.1. *Let S be an LCM-closed set of distinct positive integers, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. Then*

$$\det(S)_f = \prod_{v=1}^n f(x_v)^2 \sum_{\substack{x_v | z | \text{lcm } S \\ x_t \uparrow z \\ v < t}} \left[\zeta * \left(\frac{\mu}{f_z} \right) \right] \left(\frac{\text{lcm } S}{z} \right). \quad (6.12)$$

Corollary 6.2. *Let S be a set of distinct positive integers which is multiple-closed up to $\text{lcm } S$, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. Then*

$$\det(S)_f = \prod_{v=1}^n f(x_v)^2 \left[\zeta * \left(\frac{\mu}{f_{x_v}} \right) \right] \left(\frac{\text{lcm } S}{x_v} \right). \quad (6.13)$$

We next derive formulas for inverses of meet matrices. In Theorem 6.5 we give an expression for the inverse of $(X, Y)_f$ on arbitrary sets X and Y , and in Theorem 6.6 we give an expression for the inverse of $(S)_f$ on arbitrary set S . Taking $(P, \preceq) = (\mathbb{Z}^+, |)$ we could obtain a formula for the inverse of GCD matrices on arbitrary set S . In Theorems 6.7 and 6.8, respectively, we calculate the inverse of $(S)_f$ in cases when S is join-closed and upper-closed up to $\vee S$. Formulas similar to Theorems 6.7 and 6.8, although with stronger assumptions, have been presented earlier in [18].

Theorem 6.5. *Let $X_i = X \setminus \{x_i\}$ and $Y_i = Y \setminus \{y_i\}$ for $i = 1, 2, \dots, n$. If $[X, Y]_f$ is invertible, then the inverse of $(X, Y)_f$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$\begin{aligned} b_{ij} &= \frac{(-1)^{i+j}}{f(x_j)f(y_i)\det(X, Y)_f} \left(\prod_{v=1}^n f(x_v)f(y_v) \right) \\ &\times \left(\sum_{1 \leq k_1 < k_2 < \dots < k_{n-1} \leq m} \det E(X_j)_{(k_1, k_2, \dots, k_{n-1})} \det E(Y_i)_{(k_1, k_2, \dots, k_{n-1})} \right. \\ &\left. \times \Psi_{D, 1/f}(d_{k_1}) \Psi_{D, 1/f}(d_{k_2}) \cdots \Psi_{D, 1/f}(d_{k_{n-1}}) \right). \end{aligned} \quad (6.14)$$

Theorem 6.6. *Let $S_i = S \setminus \{x_i\}$ for $i = 1, 2, \dots, n$. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$ with*

$$\begin{aligned} b_{ij} &= \frac{(-1)^{i+j}}{f(x_i)f(x_j)\det(S)_f} \left(\prod_{v=1}^n f(x_v)^2 \right) \\ &\times \left(\sum_{1 \leq k_1 < k_2 < \dots < k_{n-1} \leq m} \det E(S_i)_{(k_1, k_2, \dots, k_{n-1})} \det E(S_j)_{(k_1, k_2, \dots, k_{n-1})} \right. \\ &\left. \times \Psi_{D, 1/f}(d_{k_1}) \Psi_{D, 1/f}(d_{k_2}) \cdots \Psi_{D, 1/f}(d_{k_{n-1}}) \right). \end{aligned} \quad (6.15)$$

Theorem 6.7. Suppose that S is join-closed. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$ with

$$b_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu_S(x_k, x_i)\mu_S(x_k, x_j)}{\Psi_{S,1/f}(x_k)}. \quad (6.16)$$

Here μ_S is the Möbius function of the poset (S, \preceq) .

Example 6.3. By Theorem 6.7, the inverse of the MIN matrix $(S)_f$ in Example 6.1 is the $n \times n$ tridiagonal matrix $B = (b_{ij})$ with

$$b_{ij} = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \frac{1}{x_2 - x_1} \frac{x_2 + t}{x_1 + t} & \text{if } i = j = 1, \\ \frac{1}{x_i + t} \left(\frac{x_{i-1} + t}{x_i - x_{i-1}} + \frac{x_{i+1} + t}{x_{i+1} - x_i} \right) & \text{if } 1 < i = j < n, \\ \frac{1}{x_n + t} \left(\frac{x_{n-1} + t}{x_n - x_{n-1}} + 1 \right) & \text{if } i = j = n, \\ \frac{-1}{|x_i - x_j|} & \text{if } |i - j| = 1. \end{cases}$$

Theorem 6.8. Suppose that S is upper-closed up to $\vee S$. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \preceq x_i \wedge x_j} \frac{\mu(x_k, x_i)\mu(x_k, x_j)}{\Psi_{S,1/f}(x_k)}. \quad (6.17)$$

Here μ is the Möbius function of (P, \preceq) .

Example 6.4. By Theorem 6.8, the inverse of the MIN matrix $(S)_f$ in Example 6.2 is the $n \times n$ tridiagonal matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 0 & \text{if } |i - j| > 1, \\ \frac{x_1 + 1 + t}{x_1 + t} & \text{if } i = j = 1, \\ 2 & \text{if } 1 < i = j < n, \\ 1 & \text{if } i = j = n, \\ -1 & \text{if } |i - j| = 1. \end{cases}$$

Corollary 6.3. Let S be a set of distinct positive integers which is multiple-closed up to $\text{lcm } S$, and let f be a quasimultiplicative arithmetical function such that $f(r) \neq 0$ for all $r \in \mathbb{Z}^+$. If the GCD matrix $(S)_f$ is invertible, then its inverse is the $n \times n$ matrix $B = (b_{ij})$, where

$$b_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k | (x_i, x_j)} \frac{\mu(x_i/x_k)\mu(x_j/x_k)}{\left[\zeta * \left(\frac{\mu}{f_{x_k}} \right) \right] \left(\frac{\text{lcm } S}{x_k} \right)}. \quad (6.18)$$

Here μ is the number-theoretic Möbius function.

References

- [1] M. Aigner, Combinatorial Theory, Springer–Verlag, New York, 1979.
- [2] E. Altinisik, On inverses of GCD matrices associated with multiplicative functions and a proof of the Hong-Loewy conjecture, *Linear Algebra Appl.* 430 (2009) 1313–1327.
- [3] E. Altinisik, B. E. Sagan and N. Tuglu, GCD matrices, posets, and nonintersecting paths, *Linear Multilinear Algebra* 53 (2005) 75–84.
- [4] E. Altinisik, N. Tuglu and P. Haukkanen, Determinant and Inverse of Meet and Join Matrices, *Int. J. Math. Math. Sci.* vol. 2007 (2007) Article ID 37580.
- [5] S. Beslin and S. Ligh, Greatest common divisor matrices, *Linear Algebra Appl.* 118 (1989) 69–76.
- [6] R. Bhatia and J. A. Dias da Silva, Infinite divisibility of GCD matrices, *Amer. Math. Monthly* 115 (2008) 551–553.
- [7] K. Bourque and S. Ligh, Matrices associated with multiplicative functions, *Linear Algebra Appl.* 216 (1995) 267–275.
- [8] P. Haukkanen, On meet matrices on posets, *Linear Algebra Appl.* 249 (1996) 111–123.
- [9] P. Haukkanen and J. Sillanpää, Some analogues of Smith’s determinant, *Linear Multilinear Algebra* 41 (1996) 233–244.
- [10] P. Haukkanen, J. Wang and J. Sillanpää, On Smith’s determinant, *Linear Algebra Appl.* 258 (1997) 251–269.
- [11] S. Hong, Gcd-closed sets and determinants of matrices associated with arithmetical functions, *Acta Arith.* 101(4) (2002) 321–332.
- [12] S. Hong and K. S. Enoch Lee, Asymptotic behavior of eigenvalues of reciprocal power LCM matrices, *Glasg. Math. J.* 50 (2008) 163–174.
- [13] S. Hong and R. Loewy, Asymptotic behavior of eigenvalues of greatest common divisor matrices, *Glasg. Math. J.* 46 (2004) 551–569.
- [14] S. Hong and R. Loewy, Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod r), *Int. J. Number Theory*, accepted.
- [15] S. Hong and Q. Sun, Determinants of matrices associated with incidence functions on posets, *Czechoslovak Math. J.* 54 (2004) 431–443.
- [16] P. Ilmonen, P. Haukkanen and J.K. Merikoski, On eigenvalues of meet and join matrices associated with incidence functions, *Linear Algebra Appl.* 429 (2008) 859–874.

- [17] I. Korkee and P. Haukkanen, Bounds for determinants of meet matrices associated with incidence functions, *Linear Algebra Appl.* 329 (2001) 77–88.
- [18] I. Korkee and P. Haukkanen, On meet and join matrices associated with incidence functions, *Linear Algebra Appl.* 372 (2003) 127–153.
- [19] I. Korkee and P. Haukkanen, On meet matrices with respect to reduced, extended and exchanged sets, *JP J. Algebra Number Theory Appl.* 4(3) (2004), 559–575.
- [20] I. Korkee and P. Haukkanen, On a general form of meet matrices associated with incidence functions, *Linear Multilinear Algebra* 53 (2005), 309–321.
- [21] I. Korkee and P. Haukkanen, On a general form of join matrices associated with incidence functions, *Aequationes Math.* 75 (2008) no. 1-2, 29–42.
- [22] Z. Li, The determinants of GCD matrices, *Linear Algebra Appl.* 134 (1990) 137–143.
- [23] J.-G. Luque, Hyperdeterminants on semilattices, *Linear Multilinear Algebra* 56 (2008) 333–344.
- [24] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer–Verlag, New York, 1986.
- [25] J. S. Owall, An analysis of GCD and LCM matrices via the LDL^T -factorization, *Electron. J. Linear Algebra* 11 (2004) 51–58.
- [26] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis, Band II*, 4. Auflage, Springer–Verlag, Berlin–Heidelberg–New York, 1971.
- [27] B. V. Rajarama Bhat, On greatest common divisor matrices and their applications, *Linear Algebra Appl.* 158 (1991) 77–97.
- [28] J. Sándor and B. Crstici, *Handbook of Number Theory II*, Kluwer Academic Publishers, 2004.
- [29] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Marcel Dekker, New York, 1989.
- [30] H. J. S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc.* 7 (1875/76) 208–212.
- [31] R. P. Stanley, *Enumerative Combinatorics. Vol. 1*. Corrected reprint of the 1986 original. Cambridge studies in Advanced Mathematics, 49. Cambridge University Press, 1997.